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On reduction properties in the theory of lubrication

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SUMMARY

In the case of constant film thickness, Reynolds' equation of plane lubrication theory reduces to Laplace's equation for the pressure distribution. In this paper, non-constant forms for the film thickness are constructed for which reduction of the two-dimensional problem to consideration of Laplace's equation is possible. This is achieved via Baecklund-type transformations; the approach is somewhat analogous to one adopted in gasdynamics and recently in other areas of Continuum Mechanics to obtain canonical forms for systems descriptive of physical situations.

1. Introduction

Baecklund-type transformations have been employed in many physical contexts in recent years. Ames [1] gives an introductory account and discusses applications to the theory of propagation of optical pulses. A generalization of the concept of Baecklund transformation has been utilized in gasdynamics to reduce the hodograph equations to canonical form in subsonic, transonic and supersonic flow (Loewner [2], Power, Rogers and Osborn [3]). A similar approach may be adopted for hodograph equations descriptive of the propagation of large amplitude disturbances in non-linear elastic media. Thus, reduction of the hyperbolic system to an associated wave equation may be sought (Rogers [4]). This reduction may be achieved for certain multi-parameter non-linear stress-strain laws recently introduced by Cekirge and Varley [5] in another manner. It may also be shown that Weinstein's correspondence principle can be generated as a particular Baecklund transformation of the Stokes–Beltrami equations (Clements and Rogers [6]). An iterated form of the correspondence principle may be used to solve certain boundary-value problems involving axially-symmetric deformations of incompressible isotropic linear elastic materials with solid inclusions. Other applications of generalized Baecklund transformations have been made for example in elastic-plastic wave propagation (Rogers and Clements [7]) and wave propagation through inhomogeneous elastic media (Clements and Rogers [8]). The present paper establishes new applications in plane lubrication theory. In particular, Weinstein's correspondence principle is iterated to generate the solution of Reynolds' equation for certain three-parameter forms of the film thickness.

2. Reynolds' equation

Under the usual assumptions of hydrodynamic lubrication concerning the flow of a thin film of incompressible oil between two neighbouring surfaces in relative tangential motion, the pressure distribution within the film satisfies Reynolds' equation

$$\frac{\partial}{\partial x}\left(h^3\mu^{-1}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(h^3\mu^{-1}\frac{\partial p}{\partial y}\right)=6U\frac{dh}{dx}, \tag{2.1}$$

where μ is the oil viscosity, h is the film thickness at any point and U is the speed of the moving surface in the direction of the x-axis.

Introduction of the non-dimensional variables

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$$\bar{x} = \frac{x}{l}, \quad \bar{y} = \frac{y}{l}, \quad \bar{h} = \frac{h}{h_0}, \quad \bar{\pi} = \frac{ph_0^2}{12\mu Ul} \tag{2.2}$$

where l and h_0 are representative lengths reduces Reynolds' equation to (if μ is assumed constant)

$$\frac{\partial}{\partial \bar{x}}\left(\bar{h}^3 \frac{\partial \bar{\pi}}{\partial \bar{x}}\right) + \frac{\partial}{\partial \bar{y}}\left(\bar{h}^3 \frac{\partial \bar{\pi}}{\partial \bar{y}}\right) = \frac{1}{2}\frac{d\bar{h}}{d\bar{x}}. \tag{2.3}$$

If the surfaces are parallel, as is the case for many hydrostatic bearings, then $\bar{h}$ = constant and (2.3) becomes Laplace's equation. However, when the film thickness is variable, it is of interest to determine conditions under which (2.3) may be transformed to Laplace's equation. Here it is shown that such a reduction is possible for a wide class of forms for $\bar{h}(\bar{x})$. Moreover, the forms involve arbitrary parameters available for approximation purposes. Methods analogous to those adopted here have proved of great importance in, for example, gasdynamics; they have led to such important concepts as the Kármán–Tsien approximation of subsonic flow. In that instance, however, the reduction to Laplace's equation was in the hodograph plane. Here, there is no such disadvantage, the reduction being achieved in the physical plane.

Before proceeding to the transformations, it is convenient to represent the lubrication equations in a matrix form. Thus, if $\bar{p}$ is introduced according to

$$\bar{p} = \bar{\pi} - \tfrac{1}{2}\int \bar{h}^{-2} d\bar{x}, \tag{2.4}$$

equation (2.3) reduces to

$$\frac{\partial}{\partial \bar{x}}\left(\bar{h}^3 \frac{\partial \bar{p}}{\partial \bar{x}}\right) + \frac{\partial}{\partial \bar{y}}\left(\bar{h}^3 \frac{\partial \bar{p}}{\partial \bar{y}}\right) = 0. \tag{2.5}$$

The latter may, in turn, be identically satisfied by introducing $\bar{\phi}(\bar{x}, \bar{y})$ defined via the matrix equation

$$\begin{pmatrix} \bar{\phi} \\ \bar{p} \end{pmatrix}_{\bar{x}} = \begin{pmatrix} 0 & \bar{h}^3 \\ -\bar{h}^{-3} & 0 \end{pmatrix} \begin{pmatrix} \bar{\phi} \\ \bar{p} \end{pmatrix}_{\bar{y}}. \tag{2.6}$$

This provides a convenient form for the application of Baecklund-type transformations. Specifically, matrix transformations are introduced in the subsequent section with a view to the reduction of (2.6) to a form associated with the Cauchy–Riemann equations, namely

$$\begin{pmatrix} \phi^* \\ p^* \end{pmatrix}_{x^*} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \phi^* \\ p^* \end{pmatrix}_{y^*}. \tag{2.7}$$

3. The matrix transformations

The system (2.6) may be written in the form

$$\bar{\Omega}_{\bar{x}} = \bar{H}\bar{\Omega}_{\bar{y}}, \tag{3.1}$$

where

$$\bar{\Omega} = \begin{pmatrix} \bar{\phi} \\ \bar{p} \end{pmatrix}, \quad \bar{H} = \begin{pmatrix} 0 & \bar{h}^3 \\ -\bar{h}^{-3} & 0 \end{pmatrix}. \tag{3.2), (3.3}$$

Matrix transformations of the type

$$\left.\begin{aligned} &\Omega^*_{x^*} = A_1 \bar{\Omega}_{\bar{x}} + B_1 \bar{\Omega}, \quad |A_1| \neq 0, \\ &\Omega^*_{y^*} = A_2 \bar{\Omega}_{\bar{y}} + B_2 \bar{\Omega}, \quad |A_2| \neq 0 \\ &x^* = \bar{x}, \quad y^* = \bar{y} \end{aligned}\right\} \tag{3.4}$$

are sought which transform the system (3.1)–(3.3) to the elliptic canonical form (2.7), that is

$$\Omega^*_{x^*} = H^* \Omega^*_{y^*}, \tag{3.5}$$

where

$$\Omega^* = \begin{pmatrix} \phi^* \\ p^* \end{pmatrix}, \qquad H^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (3.6), (3.7)$$

In (3.4), A_j, B_j, $j=1, 2$ are 2×2 matrices with entries arbitrary functions of the variables $\bar{x}$, $\bar{y}$.

Imposing the commutativity conditions

$$\bar{\Omega}_{\bar{x}\bar{y}} = \bar{\Omega}_{\bar{y}\bar{x}}, \qquad \Omega^*_{x^*y^*} = \Omega^*_{y^*x^*} \qquad (3.8), (3.9)$$

on the relations (3.4), it is seen that

$$(A_1 - A_2)\bar{\Omega}_{\bar{x}\bar{y}} + (A_{1,\bar{y}} - B_2)\bar{\Omega}_{\bar{x}} + (B_1 - A_{2,\bar{x}})\bar{\Omega}_{\bar{y}} + (B_{1,\bar{y}} - B_{2,\bar{x}})\bar{\Omega} = 0. \tag{3.10}$$

In view of (3.1), the matrix equation (3.10) is identically satisfied by setting

$$A_1 = A_2, \tag{3.11}$$

$$(A_{1,\bar{y}} - B_2)\bar{H} + B_1 - A_{2,\bar{x}} = 0, \tag{3.12}$$

$$B_{1,\bar{y}} - B_{2,\bar{x}} = 0. \tag{3.13}$$

Further, from (3.4) it follows that

$$\Omega^*_{x^*} - H^* \Omega^*_{y^*} = A_1 [\bar{\Omega}_{\bar{x}} - A_1^{-1} H^* A_1 \bar{\Omega}_{\bar{y}}] + (B_1 - H^* B_2)\bar{\Omega},$$

so that, setting

$$A_1^{-1} H^* A_1 = \bar{H}, \qquad B_1 = H^* B_2, \qquad (3.14), (3.15)$$

the system $\bar{\Omega}_{\bar{x}} = \bar{H}\bar{\Omega}_{\bar{y}}$ is transformed to the associated system $\Omega^*_{x^*} = H^* \Omega^*_{y^*}$ and conversely by the transformations defined by (3.4). Thus, summarizing, it is seen that

$$\bar{\Omega}_{\bar{x}} = \bar{H}\bar{\Omega}_{\bar{y}} \leftrightarrow \Omega^*_{x^*} = H^* \Omega^*_{y^*} \tag{3.16}$$

via the matrix Baecklund-type transformations defined by (3.4) subject to the conditions (3.11)–(3.15). In particular, if $A_1 = A_2$ and B_1, B_2 are taken to be independent of $\bar{y}$, (3.13) and (3.15) indicate that B_1 and B_2 are necessarily constant matrices, while (3.12) reduces to

$$A_{1,\bar{x}} - H^* B_2 + B_2 A_1^{-1} H^* A_1 = 0. \tag{3.17}$$

Moreover, the property of zero principal diagonal elements is preserved under the mapping $\bar{H} \to H^*$ if (but not only if) A_1 assumes the diagonal form

$$A_1 = \begin{pmatrix} a_1^1 & 0 \\ 0 & a_2^2 \end{pmatrix}, \tag{3.18}$$

in which case

$$H^* = A_1 \bar{H} A_1^{-1} = \begin{pmatrix} 0 & a_1^1 \bar{h}_2^1 / a_2^2 \\ a_2^2 \bar{h}_1^2 / a_1^1 & 0 \end{pmatrix}, \quad (\bar{H} \equiv [\bar{h}_j^i]). \tag{3.19}$$

Inspection of (3.17) shows that B_2 is necessarily of the form

$$B_2 = \begin{pmatrix} 0 & b_2^1 \\ b_1^2 & 0 \end{pmatrix}, \tag{3.20}$$

and the matrix equation (3.17) yields

$$(a_1^1)_{\bar{x}} - h_2^{1*} b_1^2 + h_1^{2*} b_2^1 (a_1^1/a_2^2) = 0, \tag{3.21}$$

$$(a_2^2)_{\bar{x}} - h_1^{2*} b_2^1 + h_2^{1*} b_1^2 (a_2^2/a_1^1) = 0. \tag{3.22}$$

The latter pair of equations combine to show that

$$\text{Det } A_1 = a_1^1 a_2^2 = \text{constant} = \lambda, \quad \lambda \neq 0 \tag{3.23}$$

so that the system (3.21), (3.22) may be reduced to (3.23) together with a single Riccati-type equation in either a_1^1 or a_2^2. In particular, the Riccati equation in a_1^1 is, on setting $h_2^{1*} = +1$, $h_1^{2*} = -1$,

$$(a_1^1)_{\bar{x}} + \alpha (a_1^1)^2 + \beta = 0\,, \quad (\alpha = -b_2^1 \lambda^{-1}\,,\ \beta = -b_1^2)\,.$$

Hence,

(a) if $\alpha = 0$: $a_1^1 = -\beta\bar{x} + \delta$

(b) if $\beta = 0$: $a_1^1 = 1/(\alpha\bar{x} + \varepsilon)$

(c) if $\beta/\alpha > 0$: $a_1^1 = (\beta/\alpha)^{\frac{1}{2}} \cot\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x} + \xi)\}$

(d) if $\beta/\alpha < 0$: $a_1^1 = (-\beta/\alpha)^{\frac{1}{2}} \tanh\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x} + \xi)\}$

where δ, ε, ξ, η are arbitrary constants of integration.

Now, (3.19) shows that

$$\bar{h}(\bar{x}) = \lambda^{\frac{1}{3}} (a_1^1)^{-\frac{2}{3}} \tag{3.24}$$

whence, it is concluded that reduction of the Reynolds' system defined by (3.1)–(3.3) may be achieved when $\bar{h}(\bar{x})$ adopts one of the forms

(a) $\lambda^{\frac{1}{3}}[-\beta\bar{x} + \delta]^{-\frac{2}{3}}$

(b) $\lambda^{\frac{1}{3}}[\alpha\bar{x} + \varepsilon]^{\frac{2}{3}}$

(c) $\lambda^{\frac{1}{3}}(\alpha/\beta)^{\frac{1}{3}} \tan^{\frac{2}{3}}\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x} + \xi)\}$

(d) $\lambda^{\frac{1}{3}}(-\alpha/\beta)^{\frac{1}{3}} \coth^{\frac{2}{3}}\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x} + \xi)\}$.

In fact, (a)–(d) are not the only film thickness forms for which Reynolds' equation may be reduced to Laplace's equation via Baecklund transformations. In particular, in the next section, it is noted how repeated application of such transformations can lead to new results.

4. Iterated Baecklund transformations

In a recent paper on Baecklund-type transformations of the Stokes–Beltrami equations (Rogers and Kingston [9]) it was demonstrated that matrix systems

$$\bar{\Omega}_{\bar{x}} = \bar{H}\bar{\Omega}_{\bar{y}}\,, \quad \bar{H} = \begin{pmatrix} 0 & \bar{x}^s \\ -\bar{x}^{-s} & 0 \end{pmatrix}, \quad (s \in R) \tag{4.1}$$

may be linked to associated systems

$$\Omega^*_{x^*} = H^*\Omega^*_{y^*}\,, \quad H^* = \begin{pmatrix} 0 & x^{*t} \\ -x^{*-t} & 0 \end{pmatrix}, \quad (t \in R) \tag{4.2}$$

in the six cases

(i) $s = t$, (ii) $s = -t$, (iii) $s = t-2$, $s \neq -1$

(iv) $s = t+2$, $s \neq 1$ (v) $s = -t-2$, $s \neq -1$ (vi) $s = -t+2$, $s \neq 1$.

In particular, the invariant transformations (i) contain a result due to Parsons [10] as a special case. Further, (iii) generates a four-parameter class of correspondence principles extending Weinstein's correspondence principle (see [6]). Weinstein's principle has had a number of interesting physical applications, notably to the theory of shafts of revolution under torsion. Iterated versions of the principle have been used by Burns [11] to systematize the study of problems involving Stokes' flow of a viscous fluid past such bodies as a spindle, lens or torus and in [6] to solve boundary value problems involving rigid inclusions in incompressible elastic materials.

Consider the Baecklund-type transformations of the type (3.4) with the specializations (see [9])

$$A_1 = A_2 = \begin{pmatrix} 0 & 2d\bar{x}^{-s-1} \\ -2d\bar{x}^{s+1} & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ -2d(s+1)\bar{x}^{-s-2} & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2d(s+1) & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.3}$$

These transformations link the Stokes–Beltrami system

$$\bar{\Omega}_{\bar{x}} = \bar{H}\bar{\Omega}_{\bar{y}}, \quad \bar{H} = \begin{pmatrix} 0 & \bar{x}^{-s-2} \\ -\bar{x}^{s+2} & 0 \end{pmatrix}, \tag{4.4}$$

with the associated system

$$\Omega^*_{x^*} = H^*\Omega^*_{y^*}, \quad H^* = \begin{pmatrix} 0 & x^{*-s} \\ -x^{*s} & 0 \end{pmatrix}, \tag{4.5}$$

Explicitly the transformations yield

$$\begin{aligned} &\phi^*_{x^*} = 2d\bar{x}^{-s-1}\bar{p}_{\bar{x}}, \quad \phi^*_{y^*} = 2d\bar{x}^{-s-1}\ \bar{p}_y + 2d(s+1)\bar{\phi}, \\ &p^*_{x^*} = -2d\bar{x}^{s+1}\bar{\phi}_{\bar{x}} - 2d(s+1)\bar{x}^s\bar{\phi}, \quad p^*_{y^*} = -2d\bar{x}^{s+1}\bar{\phi}_{\bar{y}} \\ &\qquad (x^* = \bar{x}, \ y^* = \bar{y}) \end{aligned} \tag{4.6}$$

The latter pair of relations give that

$$p^* = -2d\bar{x}^{s+1}\bar{\phi}, \tag{4.7}$$

or, in the notation of Weinstein [12],

$$\bar{p}\{s\} = C\bar{x}^{s+1}\bar{\phi}\{s+2\}, \quad (C \equiv -2d). \tag{4.8}$$

The important relation (4.8) is known as Weinstein's correspondence principle; its significance lies in its use in the simplification of certain boundary value problems (see [6], [11]). It has been proved by Kingston [13] that by repeated application of the Baecklund transformation (4.6) with $2d = +1$, the general solution of the system (4.1) with $s = 2N$ ($N = 0, 1, 2, \ldots$) may be obtained in the form

$$-\bar{\phi} + i\bar{x}^{-2N}\bar{p} = \sum_{r=0}^{N} \frac{(-1)^r 2^r (2N-r)!}{r!(N-r)!}\, \bar{x}^{-2N+r} \left\{ \frac{(N-r)}{(2N-r)}\, \Phi^{(r)}(\zeta) - \frac{N}{(2N-r)}\, \overline{\Phi^{(r)}(\zeta)} \right\} \equiv \Lambda(\bar{x}, \bar{y}) \tag{4.9}$$

$$\left(\frac{N}{(2N-r)} = 1, \quad \frac{(N-r)}{(2N-r)} = 0 \text{ when } r = N = 0 \right)$$

where $\Phi^{(r)}(\zeta) \equiv \partial^r \Phi(\zeta)/\partial\zeta^r$, $\Phi(\zeta)$ is an arbitrary analytic function of $\zeta = \bar{x} + i\bar{y}$ and $\overline{\Phi^{(r)}(\zeta)}$ is the complex conjugate of $\Phi^{(r)}(\zeta)$.

Hence, referring to (3.1)–(3.3), it is seen that the general solution to Reynolds' equation may be obtained when the film thickness $\bar{h}(\bar{x})$ adopts the form (on appropriate scale change)

$$[a\bar{x} + b]^{-2N/3}, \quad N = 0, 1, 2, \ldots \tag{4.10}$$

The case $N = 0$ corresponds to the usual constant film thickness assumption in which (3.1)–(3.3) becomes the Cauchy–Riemann system. The case $N = 1$ yields the form (a) of the preceding section. The cases $N = 2, 3, \ldots$ represent new forms.

Furthermore, it is observed that the simple transformation

$$\bar{\phi} \to p^\dagger, \quad \bar{p} \to -\phi^\dagger, \tag{4.11}$$

transforms the system (4.5) to the system

$$\Omega^\dagger_{x^\dagger} = H^\dagger \Omega^\dagger_{x^\dagger}, \quad H^\dagger = \begin{pmatrix} 0 & x^{\dagger s} \\ -x^{\dagger -s} & 0 \end{pmatrix}, \quad \Omega^\dagger = \begin{pmatrix} \phi^\dagger \\ \psi^\dagger \end{pmatrix}, \tag{4.12}$$

so that the solution of the latter system is given by

$$p^{\dagger}+i\bar{x}^{-2N}\phi^{\dagger}=-\sum_{r=0}^{N}\frac{(-1)^{r}2^{r}(2N-r)!}{r!(N-r)!}\bar{x}^{-2N+r}\left\{\frac{(N-r)}{(2N-r)}\Phi^{(r)}(\zeta)-\frac{N}{(2N-r)}\overline{\Phi^{(r)}(\zeta)}\right\}\equiv -\Lambda(\bar{x},\bar{y})\,. \quad (4.13)$$

This provides the general solution to Reynolds' equation when the film thickness adopts the three-parameter form

$$[a\bar{x}+b]^{2N/3}\,, \qquad N=0,1,2,\ldots \quad (4.14)$$

The case $N=1$ is form (a) of the preceding section.

5. Integration of Reynolds' equation in cases (c) and (d)

In the last section, Reynolds' equation was solved when $\bar{h}(\bar{x})$ may be approximated by either of the forms (4.10) or (4.14), the cases (a), (b) of section 3 emerging as special cases. It remains to solve Reynolds' equation for the forms (c), (d) of that section.

The transformations under consideration are

$$\left.\begin{aligned}
&\begin{pmatrix}\phi^*\\ p^*\end{pmatrix}_{x^*}=\begin{pmatrix}a_1^1 & 0\\ 0 & a_2^2\end{pmatrix}\begin{pmatrix}\bar{\phi}\\ \bar{p}\end{pmatrix}_{\bar{x}}+\begin{pmatrix}b_1^2 & 0\\ 0 & -b_2^1\end{pmatrix}\begin{pmatrix}\bar{\phi}\\ \bar{p}\end{pmatrix},\\
&\begin{pmatrix}\phi^*\\ p^*\end{pmatrix}_{y^*}=\begin{pmatrix}a_1^1 & 0\\ 0 & a_2^2\end{pmatrix}\begin{pmatrix}\bar{\phi}\\ \bar{p}\end{pmatrix}_{\bar{y}}+\begin{pmatrix}0 & b_2^1\\ b_1^2 & 0\end{pmatrix}\begin{pmatrix}\bar{\phi}\\ \bar{p}\end{pmatrix},\\
&x^*=\bar{x}\,,\quad y^*=\bar{y}
\end{aligned}\right\} \quad (5.1)$$

where a_1^1 adopts either of the forms (c), (d). These cases are now investigated.

In case (c), the system (5.1) yields

$$\left.\begin{aligned}
\phi^*_{\bar{x}}&=(\beta/\alpha)^{\frac{1}{2}}\cot\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}\bar{\phi}_{\bar{x}}+b_1^2\bar{\phi}\,,\\
\phi^*_{\bar{y}}&=(\beta/\alpha)^{\frac{1}{2}}\cot\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}\bar{\phi}_{\bar{y}}+b_2^1\bar{p}\,,\\
p^*_{\bar{x}}&=\lambda(\alpha/\beta)^{\frac{1}{2}}\tan\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}\bar{p}_{\bar{x}}-b_2^1\bar{p}\,,\\
p^*_{\bar{y}}&=\lambda(\alpha/\beta)^{\frac{1}{2}}\tan\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}\bar{p}_{\bar{y}}+b_1^2\bar{\phi}\,.
\end{aligned}\right\} \quad (5.2)$$

Now, from (2.7),

$$\phi^*=f(\zeta)+\overline{f(\zeta)}\,,\quad p^*=i[-f(\zeta)+\overline{f(\zeta)}] \quad (5.3),(5.4)$$

where f is an arbitrary analytic function of $\zeta=\bar{x}+i\bar{y}$. Hence, $(5.2)_{1,3}$ give that

$$\frac{\partial}{\partial\bar{x}}[\bar{\phi}\cos\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}]=(\alpha/\beta)^{\frac{1}{2}}[f'+\bar{f}']\sin\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\} \quad (5.5)$$

$$\frac{\partial}{\partial\bar{x}}[\bar{p}\sin\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}]=i(\beta/\alpha)^{\frac{1}{2}}\lambda^{-1}[-f'+\bar{f}']\cos\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\} \quad (5.6)$$

so that, on integration,

$$\begin{aligned}
\bar{\phi}=&(\alpha/\beta)^{\frac{1}{2}}[(f+\bar{f})\tan\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}-\\
&-(\alpha\beta)^{\frac{1}{2}}\sec\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}\textstyle\int(f+\bar{f})\cos\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}\,d\bar{x}]\\
&+\bar{Y}_1(\bar{y})\sec\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}\,,
\end{aligned} \quad (5.7)$$

$$\begin{aligned}
\bar{p}=&\,i(\beta/\alpha)^{\frac{1}{2}}\lambda^{-1}[(-f+\bar{f})\cot\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}+\\
&+(\alpha\beta)^{\frac{1}{2}}\operatorname{cosec}\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}\textstyle\int(-f+\bar{f})\sin\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\,d\bar{x}]\\
&+\bar{Y}_2(\bar{y})\operatorname{cosec}\{(\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}\,,
\end{aligned} \quad (5.8)$$

where $\bar{Y}_1$, $\bar{Y}_2$ are functions of $\bar{y}$ to be determined. Thus, employing the relations

$$\bar{\phi}_{\bar{x}} = [\lambda/(a_1^1)^2]\bar{p}_{\bar{y}}, \quad \bar{p}_{\bar{x}} = -[(a_1^1)^2/\lambda]\bar{\phi}_{\bar{y}}, \tag{5.9}$$

it is seen that

$$\bar{Y}_1 = \lambda\alpha^{\frac{1}{2}}\beta^{-\frac{3}{2}}\bar{Y}_2', \quad \bar{Y}_1' = \lambda\alpha^{\frac{3}{2}}\beta^{-\frac{1}{2}}\bar{Y}_2,$$

whence, since $\alpha\beta > 0$,

$$\bar{Y}_1 = \bar{\alpha}\cosh[(\alpha\beta)^{\frac{1}{2}}\bar{y}] + \bar{\beta}\sinh[(\alpha\beta)^{\frac{1}{2}}\bar{y}], \tag{5.10}$$

$$\bar{Y}_2 = \lambda^{-1}(\beta/\alpha)[\bar{\alpha}\sinh[(\alpha\beta)^{\frac{1}{2}}\bar{y}] + \bar{\beta}\cosh[(\alpha\beta)^{\frac{1}{2}}\bar{y}]], \tag{5.11}$$

where $\bar{\alpha}$, $\bar{\beta}$ are arbitrary real constants of integration.

In case (d), from system (5.1),

$$\left.\begin{aligned}
\phi_{\bar{x}}^{*} &= (-\beta/\alpha)^{\frac{1}{2}}\tanh\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}\bar{\phi}_{\bar{x}} + b_1^2\bar{\phi},\\
\phi_{\bar{y}}^{*} &= (-\beta/\alpha)^{\frac{1}{2}}\tanh\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}\bar{\phi}_{\bar{y}} + b_2^1\bar{p},\\
p_{\bar{x}}^{*} &= \lambda(-\alpha/\beta)^{\frac{1}{2}}\coth\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}\bar{p}_{\bar{x}} - b_2^1\bar{p},\\
p_{\bar{y}}^{*} &= \lambda(-\alpha/\beta)^{\frac{1}{2}}\coth\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}\bar{p}_{\bar{y}} + b_1^2\bar{\phi},
\end{aligned}\right\} \tag{5.12}$$

so that, in view of (5.3), (5.4), relations $(5.11)_{1,3}$ yield

$$\frac{\partial}{\partial\bar{x}}[\bar{\phi}\sinh\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}] = (-\alpha/\beta)^{\frac{1}{2}}[f'+\bar{f}']\cosh\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}, \tag{5.13}$$

$$\frac{\partial}{\partial\bar{x}}[\bar{p}\cosh\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}] = i(-\beta/\alpha)^{\frac{1}{2}}\lambda^{-1}[-f'+\bar{f}']\sinh\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}. \tag{5.14}$$

On integration, these show that

$$\begin{aligned}
\bar{\phi} = (-\alpha/\beta)^{\frac{1}{2}}[&(f+\bar{f})\coth\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\} -\\
&-(-\alpha\beta)^{\frac{1}{2}}\operatorname{cosech}\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}\int(f+\bar{f})\sinh\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}\,d\bar{x}]\\
&+\bar{Y}_3(\bar{y})\operatorname{cosech}\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\},
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
\bar{p} = i(-\beta/\alpha)^{\frac{1}{2}}\lambda^{-1}[&(-f+\bar{f})\tanh\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\} -\\
&-(-\alpha\beta)^{\frac{1}{2}}\operatorname{sech}\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}\int(-f+\bar{f})\cosh\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\}\,d\bar{x}]\\
&+\bar{Y}_4(\bar{y})\operatorname{sech}\{(-\beta/\alpha)^{\frac{1}{2}}(\alpha\bar{x}+\xi)\},
\end{aligned} \tag{5.16}$$

where $\bar{Y}_3$, $\bar{Y}_4$ are functions of $\bar{y}$ satisfying, by virtue of the relations (5.9)

$$\bar{Y}_3 = -i\lambda\alpha^{\frac{1}{2}}\beta^{-\frac{3}{2}}\bar{Y}_4', \quad \bar{Y}_3' = -i\lambda\alpha^{\frac{3}{2}}\beta^{-\frac{1}{2}}\bar{Y}_4.$$

Thus, since $\alpha\beta < 0$ in this instance,

$$\bar{Y}_3 = \bar{\gamma}\cos[(-\alpha\beta)^{\frac{1}{2}}\bar{y}] + \bar{\delta}\sin[(-\alpha\beta)^{\frac{1}{2}}\bar{y}], \tag{5.17}$$

$$\bar{Y}_4 = \lambda^{-1}(\beta/\alpha)[\bar{\gamma}\sin[(-\alpha\beta)^{\frac{1}{2}}\bar{y}] - \bar{\delta}\cos[(-\alpha\beta)^{\frac{1}{2}}\bar{y}]], \tag{5.18}$$

where $\bar{\gamma}$, $\bar{\delta}$ are further real constants of integration. This completes the integration of Reynolds' equation when the film thickness may be approximated by a form corresponding to case (c) or (d).

6. Reduction in gasbearing theory

The work of the preceding sections has been concerned with a Reynolds' equation for an incompressible lubricant of constant viscosity. It is of importance to enquire as to whether any kind of reduction is available for the more realistic situation in which compressibility and variable viscosity effects are present; an indication of a possible approach is given below.

If the density ρ and viscosity μ of the lubricant are variable, the two-dimensional Reynolds' equation becomes, under steady conditions,

$$\frac{\partial}{\partial x}\left(\rho h^3 \mu^{-1}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\rho h^3 \mu^{-1}\frac{\partial p}{\partial y}\right)=\frac{6\partial}{\partial x}(\rho U h). \tag{6.1}$$

If we set

$$u = 6\mu U - h^2\frac{\partial p}{\partial x}, \qquad v=-h^2\frac{\partial p}{\partial y}, \qquad (6.2), (6.3)$$

it is seen that (6.1) allows the introduction of a stream function $\psi(x, y)$ according to

$$u=(\rho h)^{-1}\mu\frac{\partial \psi}{\partial y}, \qquad v=-(\rho h)^{-1}\mu\frac{\partial \psi}{\partial x}, \qquad (6.4), (6.5)$$

where, from (6.1), ψ satisfies

$$\frac{\partial}{\partial x}\left(h^{-3}\frac{\partial \psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(h^{-3}\frac{\partial \psi}{\partial y}\right)=0. \tag{6.6}$$

The latter equation is amenable to the techniques presented earlier. Thus, it may be reduced to Laplace's equation by means of Baecklund-type transformations subject to the film thickness being approximated by certain forms; of course, the same analysis is in particular, possible for the case of incompressible constant viscosity lubricant.

If, for example, the viscosity of the gas lubricant is characterized by a specified $\mu(p)$ relationship and the gas law is of the form $\rho=\rho(p)$, once the general solution for $\psi(x, y)$ corresponding to a form $h(x)$ has been obtained from the reduction of (6.6), the problem becomes that of determining p from (6.2)–(6.5) satisfying the prescribed boundary conditions. In general, the pressure will be prescribed along known geometric boundaries (see Milne [14]). The solution of specific boundary-value problems utilizing the methods presented here will be the subject of a future paper.

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